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INVERSION PROPERTY OF THE FUNDAMENTAL MATRIX ASSOCIATED WITH TRAJECTORY PERTURBATION PROBLEMS

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ABSTRACT

Systems described by ordinary linear differential equations with time-varying coefficients may be conveniently analyzed using the concepts of state variables and fundamental matrix. Characteristically, the inverse of this matrix appears in the state transition equation. An inversion property of the fundamental matrix applicable to a class of dynamic systems, which includes as a member trajectory perturbation problems, is presented. This property allows the inverse matrix to be obtained by a simple rearrangement of elements of the original matrix. When the matrix is of high order, significant advantages accrue in both time saving and numerical accuracy.

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INVERSION PROPERTY OF THE FUNDAMENTAL MATRIX ASSOCIATED WITH TRAJECTORY PERTURBATION PROBLEMS

By Alan L. Friedlander[1]

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Increased emphasis has been given recently to the application of linear perturbation techniques in studies of trajectory and guidance problems (e.g., 1, 2, 3).[2] The resulting perturbed equations of motion are given by a set of ordinary linear differential equations with time-varying coefficients. The solution of such a set can be greatly facilitated by the concepts of "state variables" and "fundamental matrices", where the state transition equations are expressed in terms of these computable matrices. Characteristically, the inverse of the fundamental matrix appears in the equations. It is recognized that inversion of high-order matrices can be both time consuming and inaccurate even with the aid of digital computers. Fortunately, in the case of perturbed trajectories there exists an inversion property, which allows the inverse to be obtained by a simple rearrangement of elements of the original matrix. Such a property has been indicated by McLean et al (2) for the special case of coasting trajectories. The purpose of the present paper is to extend the inversion property to a class of dynamic systems, which includes as a member trajectory motion influenced by an acceleration forcing function (e.g., thrust acceleration) in addition to gravitational acceleration. Also, it is felt that the usefulness of the inversion property deserves wider attention.

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[2]Numbers in parentheses indicate References at end of paper.

State Equations and Fundamental Matrix

Consider a linear system described by a set of n first-order differential equations. In vector and matrix notation

$$\frac{ds}{dt} - A(t)s(t) = B(t)f(t) \qquad [1]$$

where s is an n-dimensional state vector, f is an m-dimensional vector of forcing inputs applied to the system, and A and B are $(n \times n)$ and $(n \times m)$ coefficient matrices, respectively. The state is defined as a set of output variables from which the entire future behavior of the system may be determined, provided the future inputs to the system are known. Assume initialization of the problem at a fixed time t_o with corresponding state $s(t_o)$. In general, two types of problems are admitted; one where the region of interest lies between fixed-time interval (t_o, t_f), and the other where a terminal t_f is not specified. In either case the solution of [1] may be facilitated by introducing an $(n \times n)$ fundamental matrix $\Lambda(t)$, which satisfies the following equation

$$\frac{d\Lambda}{dt} + \Lambda(t)A(t) = 0 \qquad [2]$$

and is subject to an arbitrary boundary condition to be discussed presently. In the literature, [2] has often been called the adjoint equation to [1] and Λ is the adjoint matrix.

Premultiplying [1] by Λ, postmultiplying [2] by s, and adding the two modified equations yield

$$\frac{d}{dt}(\Lambda s) = \Lambda(t)B(t)f(t)$$

When this equation is integrated between the limits t_1 and t_2, the general state transition equation is

$$s(t_2) = \Lambda^{-1}(t_2)\Lambda(t_1)s(t_1) + \Lambda^{-1}(t_2)\int_{t_1}^{t_2} \Lambda(t)B(t)f(t)dt \qquad [3]$$

Nonsingularity of Λ is assumed, and the superscript -1 denotes the matrix inverse operation. Several interpretations of this equation are:

(1) Suppose the problem-definition does not specify a fixed terminal time. A convenient choice of boundary condition for [2] is $\Lambda(t_o) = I$ (identity matrix). Letting $t_1 = t_o$ and $t_2 = t$, [3] gives the general solution for $s(t)$ in terms of the initial state and the effect of $f(t)$ over the interval (t_o, t). If A, B, and f are assumed to be known functions of time, [1] does not have to be solved repeatedly for different values of the initial state.

(2) Suppose a fixed terminal time t_f is specified and the terminal state is of primary interest. A convenient choice of boundary condition is $\Lambda(t_f) = I$, and $\Lambda(t)$ is computed by integrating [2] backwards in time. Letting $t_2 = t_f$ and $t_1 = t$, [3] gives the terminal state in terms of the instantaneous state and the effect of $f(t)$ over (t, t_f). If a desired terminal state is specified and $s(t)$ is measured, then synthesis of a control function $f(t)$ may proceed from the terminal form of [3].

(3) Consider a dynamic process that is to be repetitively controlled based on sampled measurements of the time-varying state. Assume that the measurements are contaminated by random noise, and assume that a statistical filtering and prediction procedure is employed to improve the state measurements. The deterministic prediction equation is given by [3], and may be operated on statistically.

The previous development indicates the requirement for inverting the fundamental matrix. An inversion property, which allows great simplification of this operation, is now presented for a special class of systems.

Inversion Property of the Fundamental Matrix

Consider a class of systems having the following restrictions:

(1) The number of state variables is even.

(2) The system coefficient matrix A can be partitioned into 4 square submatrices each of order $n/2$ such that the diagonal submatrices are equal to the null matrix and the off diagonal submatrices are symmetrical.

A common example of an even-ordered state vector is a set of output variables and their first derivatives. If a system formulation does not meet the second restriction there may exist a transformation of variables that allows it to do so.

The fundamental matrix Λ may be partitioned into 4 square submatrices each of order $n/2$.

$$\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{bmatrix} \qquad [4]$$

It is proposed to show that if the identity matrix is chosen as a boundary condition for Λ, then

$$\Lambda^{-1} = \begin{bmatrix} \Lambda_4^T & -\Lambda_2^T \\ -\Lambda_3^T & \Lambda_1^T \end{bmatrix} \qquad [5]$$

The superscript T denotes the matrix transpose operation. Note that the inverse matrix is obtained by a simple rearrangement of elements; i.e., no addition or multiplication is necessary. A straightforward proof of the inversion property follows.

By definition

$$\Lambda\Lambda^{-1} = I$$

Differentiation of this expression and substitution from [2] yields

$$\frac{d\Lambda^{-1}}{dt} = A\Lambda^{-1} \qquad [6]$$

Now partition Λ^{-1} into 4 submatrices each of order $n/2$

$$\Lambda^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \qquad [7]$$

Also, as stated previously A may be partitioned as

$$A = \begin{array}{|c|c|}\hline 0 & M \\ \hline N & 0 \\ \hline\end{array} \; , \; \begin{array}{l} M^T = M \\ N^T = N \end{array} \qquad [8]$$

Equation [6] is given in partitioned form on substitution of [7] and [8]

$$\frac{d}{dt} \begin{array}{|c|c|}\hline P_1 & P_2 \\ \hline P_3 & P_4 \\ \hline\end{array} = \begin{array}{|c|c|}\hline MP_3 & MP_4 \\ \hline NP_1 & NP_2 \\ \hline\end{array} \qquad [9]$$

Now taking the transpose of the partitioned matrices Λ and A, using the symmetrical property of M and N, and substituting into the transpose of equation [2] gives

$$\frac{d}{dt} \begin{array}{|c|c|}\hline \Lambda_1^T & \Lambda_3^T \\ \hline \Lambda_2^T & \Lambda_4^T \\ \hline\end{array} = \begin{array}{|c|c|}\hline -N\Lambda_2^T & -N\Lambda_4^T \\ \hline -M\Lambda_1^T & -M\Lambda_3^T \\ \hline\end{array} \qquad [10]$$

Finally, a term by term comparison of [9] and [10] shows that the two differential equations are equivalent if

$$P = \begin{array}{|c|c|}\hline P_1 & P_2 \\ \hline P_3 & P_4 \\ \hline\end{array} = \begin{array}{|c|c|}\hline \Lambda_4^T & -\Lambda_2^T \\ \hline -\Lambda_3^T & \Lambda_1^T \\ \hline\end{array} \qquad [11]$$

In order that [11] be true in general the boundary conditions on each of the partitioned matrices must be the same. However, it has been assumed that the boundary condition is the identity matrix, e.g., $\Lambda(t_o) = I$. Thus, $\Lambda^{-1}(t_o) = \Lambda^T(t_o) = I$. Hence, the boundary conditions are the same and the proof of [5] is complete.

A more general inversion property can be extended to the case where the boundary condition on Λ <u>is not</u> the identity matrix. For example, if the actual quantities of interest are certain linear combinations of the terminal state variables, then $\Lambda(t_f)$ may not be chosen arbitrarily. The inversion property is derived from [10] and [11] by showing that $\frac{d}{dt}(P\Lambda) = 0$. Thus, the product of P and Λ must be constant for all time. From [4] and [11]

$$P\Lambda = \begin{bmatrix} \Lambda_4^T\Lambda_1 - \Lambda_2^T\Lambda_3 & \Lambda_4^T\Lambda_2 - \Lambda_2^T\Lambda_4 \\ \Lambda_1^T\Lambda_3 - \Lambda_3^T\Lambda_1 & \Lambda_1^T\Lambda_4 - \Lambda_3^T\Lambda_2 \end{bmatrix} = K \text{ (constant)} \tag{12}$$

Multiplying through by P^{-1} and then inverting gives

$$\Lambda^{-1} = K^{-1}P = K^{-1} \begin{bmatrix} \Lambda_4^T & -\Lambda_2^T \\ -\Lambda_3^T & \Lambda_1^T \end{bmatrix} \tag{13}$$

Note that [5] is a special case of [13] when $K = I$. The usefulness of [13] is apparent since K is obtained from [12] for any choice of boundary condition, and being constant it needs to be inverted only once. Also, if Λ is obtained by numerical integration, [12] may be used as a check on the accuracy of the integration.

Application to Trajectory Problems

Motion of a space vehicle expressed in fixed cartesian coordinates (x, y, z), which is influenced by a gravitational potential field $U(x, y, x)$ and a non-potential acceleration vector $f(t)$, may be described in component form by the set of six equations

$$\left.\begin{aligned} \frac{dv_x}{dt} &= -\frac{\partial U}{\partial x} + f_x \\ \frac{dx}{dt} &= v_x \end{aligned}\right\} \quad x \rightarrow y, z \tag{14}$$

where the partial derivatives are the components of the gradient of U and are continuous in the region of interest. The nonpotential acceleration components may be, for example, due to thrust. The vector velocity, position, and acceleration may be defined as

$$v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}; \qquad r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \qquad f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \tag{15}$$

If a known reference trajectory solution of the above equations is assumed, linear perturbation techniques may be used effectively to investigate perturbations about the reference and corrective guidance maneuvers. If the above equations are expanded about the reference in a Taylor series and all terms higher than first order are neglected, the perturbed equations of motion may be written as

$$\frac{ds}{dt} = \frac{d}{dt}\begin{bmatrix} \delta v \\ \hline \delta r \end{bmatrix} = A\begin{bmatrix} \delta v \\ \hline \delta r \end{bmatrix} + B\delta f \tag{16}$$

where the six-dimensional state vector s is defined in terms of the components of the perturbed velocity and position; the order of partitioning is arbitrary. It can be shown that the matrices A and B in partitioned form are

$$A = \left[\begin{array}{c|c} 0 & M \\ \hline I & 0 \end{array}\right]; \qquad B = \left[\begin{array}{c} I \\ \hline 0 \end{array}\right] \tag{17}$$

where M is made up of the second partials of U with respect to x, y, and z.

$$M = M(t) = -\begin{bmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x\, \partial y} & \frac{\partial^2 U}{\partial x\, \partial z} \\ \frac{\partial^2 U}{\partial x\, \partial y} & \frac{\partial^2 U}{\partial y^2} & \frac{\partial^2 U}{\partial y\, \partial z} \\ \frac{\partial^2 U}{\partial x\, \partial z} & \frac{\partial^2 U}{\partial y\, \partial z} & \frac{\partial^2 U}{\partial z^2} \end{bmatrix} \tag{18}$$

Since M is symmetric, the linearized trajectory problem falls into the category of system discussed previously, and the inversion property of fundamental matrices is applicable. It is important to point out, however, that the inversion property is not true if the problem is formulated in other than fixed cartesian coordinates.

If the forcing acceleration is due to thrust, $f(t) = F(t)/m(t)$, where the thrust force $F(t)$ and vehicle mass $m(t)$ may be subject to independent perturbations. Although it is possible to express δf in terms of δm and δF and avoid reformulation of the problem as given above, it may be desirable to treat δm as a state variable and δF as the forcing function. In this case the dimension of the state vector increases to 7, and the inversion property as given by [8] does not hold; however, a modified formula, which still allows rather simple inversion, may be developed. The auxiliary relation between mass flow rate, thrust, and jet velocity is added to equations [14], and f is replaced by F/m. Equation [16] then becomes

$$\frac{d}{dt}\begin{bmatrix}\delta v\\ \delta r\\ \delta m\end{bmatrix} = A\begin{bmatrix}\delta v\\ \delta r\\ \delta m\end{bmatrix} + B\delta F \qquad [19]$$

where the new coefficient matrices A and B are changed accordingly from [17]. Specifically, A is increased by a seventh row and column and may be partitioned as

$$A = \begin{array}{|c|c|c|}\hline 0 & M & a\\ \hline I & 0 & 0\\ \hline 0 & 0 & 0\\ \hline\end{array}\ ; \qquad a = -\frac{1}{m^2}\begin{bmatrix}F_x\\ F_y\\ F_z\end{bmatrix} \qquad [20]$$

Correspondingly, the fundamental matrix Λ is increased by a seventh row and column. Since both velocity and position state transition depend upon mass, while mass state transition depends only upon mass in a one-to-one fashion, Λ may be partitioned as

$$\Lambda = \begin{array}{|c|c|c|}\hline \Lambda_1 & \Lambda_2 & \lambda_1\\ \hline \Lambda_3 & \Lambda_4 & \lambda_2\\ \hline 0 & 0 & 1\\ \hline\end{array}\ ; \qquad \Lambda_{boundary} = I \qquad [21]$$

where λ_1 and λ_2 are each three-dimensional vectors. Proceeding as in the previous section it can be shown that Λ^{-1} is given by

$$\Lambda^{-1} = \begin{array}{|c|c|c|} \hline \Lambda_4^T & -\Lambda_2^T & P_1 \\ \hline -\Lambda_3^T & \Lambda_1^T & P_2 \\ \hline 0 & 0 & 1 \\ \hline \end{array} \qquad [22]$$

where

$$P_1 = -\Lambda_4^T\lambda_1 + \Lambda_2^T\lambda_2$$

$$P_2 = \Lambda_3^T\lambda_1 - \Lambda_1^T\lambda_2$$

Thus, the modified inversion property retains the major characteristic of simple term rearrangement although some algebra is required to obtain the elements of P_1 and P_2.

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